Flexibility and Rigidity in Steady Fluid Motion

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Joint work with Peter Constantin and Theodore D. Drivas

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ICL/UCL joint Pure Analysis and PDE, *Dec 11 2020*

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Primary objective: building a *confinement device* (e.g. tokamak, stellarator) to keep a hot plasma confined to a finite volume.

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A basic requirement for confinement is the existence of a flux function $\psi : T \to \mathbb{R}$ satisfying $B \cdot \nabla \psi = 0$, $|\nabla \psi| > 0$. Provided $|\nabla P| > 0$ the pressure is always a flux function.

Tokamaks and Axisymmetry

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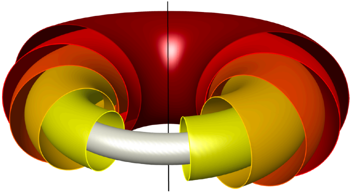
Example of “good” magnetohydrostatic equilibria are those exhibiting flux surfaces:



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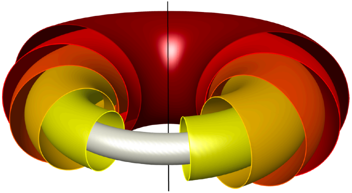

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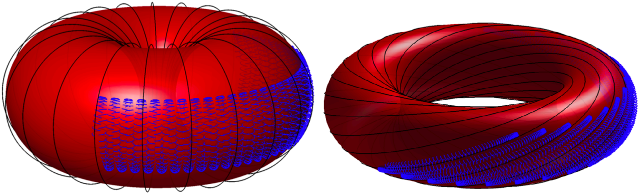
Drifts make their orbits slip off their initial field line over time.



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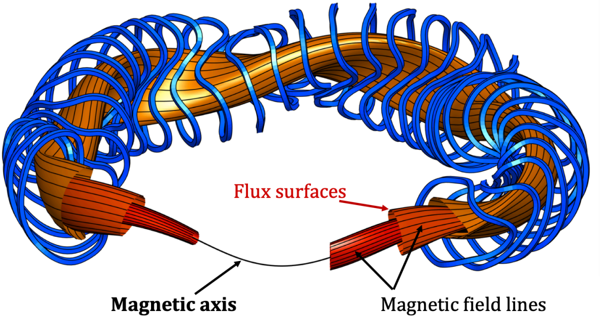

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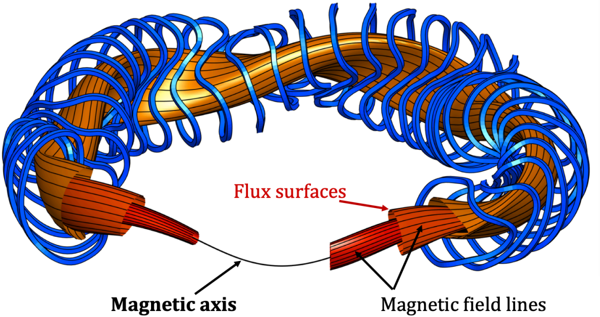

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No known examples of such an object which is an MHS equilibrium!

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H. Grad conjectured no smooth equilibria with flux functions exist outside symmetry.

Conjecture (Grad, 1967): Any non-isolated and smooth equilibrium of unforced MHS on a domain $T \subset \mathbb{R}^3$ (diffeomorphic to the solid torus) which has a pressure p possessing nested level sets foliating T is axisymmetric.

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Grad's conjecture remains open.

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The function u is determined from

$$\mathrm{div}\, J = B \cdot \nabla u + \mathrm{div}\, J^{\perp} = 0. \tag{1}$$

which becomes the magnetic differential equation

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The magnetic field B is tangent to the level sets of ψ. Pick coordinates (θ, ϕ) on each level set so that $B \cdot \nabla = \partial_\theta + \iota(\psi)\partial_\phi$. If $p = p(\psi)$ then u satisfies an equation of the form

$$(\partial_\theta + \iota(\psi)\partial_\phi)\, u = (c(\psi)\partial_\theta + d(\psi)\partial_\phi)\, f$$

Existence outside of symmetry?

Writing $u(\psi, \theta, \phi) = \sum_{m,n \in \mathbb{Z}} \widehat{u}_{mn}(\psi) e^{im\theta + in\phi}$, we have

$$(m + \iota(\psi) n) \hat{u}_{mn}(\psi) = \widehat{(c(\psi) m + d(\psi) n) f}_{mn}$$

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$$(m + \iota(\psi) n) \hat{u}_{mn}(\psi) = (c(\psi) m + d(\psi) n) \widehat{f}_{mn}$$

If u is smooth and $\iota(\psi)$ is nonconstant then the only possibility is that

whenever $m + \iota(\psi) n = 0,$ we have either $\hat{u}_{mn} = 0$ or $c(\psi) m + d(\psi) n = 0.$

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QUESTION: In what sense are fluid solutions rigid (forced to conform to spatial symmetries) or flexible (can be deformed to nearby solutions which break symmetry). We address these questions first in 2d Euler.

Rigidity for 2d Euler

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Let $D \subset \mathbb{R}^2$. The stationary two dimensional Euler equations read

$$\begin{aligned} u \cdot \nabla u &= -\nabla p, && \text{in } D, \\ \nabla \cdot u &= 0, && \text{in } D, \\ u \cdot \hat{n} &= 0, && \text{on } \partial D. \end{aligned}$$

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Fixed boundary analogue of Grad's conjecture.

Large class of steady states:

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The velocity $u = \nabla^\perp \psi$ is a solution of the Euler equation with $\omega = \operatorname{curl} u = F(\psi)$. An important subclass of solutions are Arnol'd stable. They require either

$$-\lambda_1 < F'(\psi) < 0, \qquad \text{or} \qquad 0 < F'(\psi) < \infty$$

where $\lambda_1 := \lambda_1(D) > 0$ is the smallest eigenvalue of $-\Delta$ in D.

Stability as a mechanism for rigidity.

THEOREM: Let (M, g) be a compact two-dimensional Riemannian manifold with smooth boundary ∂M and let ξ be a Killing field for g tangent to ∂M. Let $u \in C^2(M)$ be an Arnol'd stable state. Then $\mathcal{L}_\xi u = 0$.

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With $\boldsymbol{u} = \nabla_g^\perp \psi$, differentiate $\Delta_g \psi = F(\psi)$ to obtain the equation

$$\begin{aligned} \Big(\Delta_g - F'(\psi)\Big)\mathcal{L}_\xi \psi &= 0, \qquad \text{in } M, \\ \mathcal{L}_\xi \psi &= 0, \quad \text{on } \partial M. \end{aligned}$$

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Consequences: all Arnol'd stable stationary solutions are

- shears $u = v(y_2)e_{y_1}$ on the periodic channel
- radial $u = v(r)e_\theta$ on the disk (or annulus)
- non-existent on manifolds without boundary with two transverse Killing fields e.g. the two-torus or the sphere.

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Arnol'd stability is a mechanism for rigidity. Are there others?

Rigidity for 2d Euler

If the domain D_0 is a periodic channel

$$D_0 = \{(y_1, y_2) \mid y_1 \in \mathbb{T},\ y_2 \in [0,1]\},$$

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> **THEOREM:** (Hamel & Nadirashvili, 2017) Let D_0 be a periodic channel and $u_0 : D_0 \to \mathbb{R}^2$ be a $C^2(D_0)$ be solution of Euler with $\inf_{D_0} u_0 > 0$. Then u_0 is a shear, namely $u_0(y_1, y_2) = (v(y_2), 0)$ for some scalar function $v(y_2)$.

Coti-Zelati, Elgindi, Widmayer (2020) prove similar statement for Poiseuille & Kolmogorov flows. Gómez-Serrano, Park, Shi, Yao (2020) for signed vorticity.

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We generalize N&H theorem to encompass other systems. Proved in two parts.

(a) If $\psi \in C^1$ with $\nabla\psi \neq 0$ and $g \in C^1$ satisfies $\nabla^\perp\psi \cdot \nabla g = 0$, then there exists a G such that $g = G(\psi)$. This shows that *any* such steady state satisfies some elliptic problem of the form

$$\Delta\psi + f(y_2)\partial_{y_2}\psi + g(y_2, \psi) + h(\psi) = 0, \qquad \text{in } D_0.$$

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(b) Application of method of moving planes to show that if $g_{y_2}, f_{y_2} \geq 0$, all solutions of the above satisfy $\psi(y_1, y_2) = \psi(y_2)$.

Applications to fluid systems (Constantin-Drivas-G.)

$$\Delta\psi - y_2\Theta'(\psi) - G'(\psi) = 0.$$

THEOREM: (Boussinesq rigidity) Let D_0 be a periodic channel and suppose that $u_0 : D_0 \to \mathbb{R}^2$ and $\theta_0 : D_0 \to \mathbb{R}$ be a $C^2(D_0)$ solution with $\inf_{D_0} u_0 > 0$. Then there exists Lipschitz Θ_0 s.t. $\theta_0 = \Theta_0(\psi_0)$. If furthermore

$$\Theta_0'(\psi_0) \leq 0,$$

then u_0 is a shear, i,e, $u_0(y_1, y_2) = (v(y_2), 0)$ for some scalar function $v(y)$.

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$$\frac{\partial^2}{\partial r^2}\psi + \frac{\partial^2}{\partial z^2}\psi - \frac{1}{r}\frac{\partial}{\partial r}\psi + r^2 p'(\psi_0) - CC'(\psi_0) = 0.$$

THEOREM: (Axisymmetric Euler rigidity) Let $D = \{(r, z) \in [1/2, 1] \times \mathbb{T}\}$. Suppose $p, C : \mathbb{R} \to \mathbb{R}$ are Lipchitz functions and that $\psi : D \to \mathbb{R}$ is $C^2(D)$ solution of the Grad-Shafranov equation with $\inf_D |\nabla\psi| > 0$. If

$$p'(\psi) \geq 0,$$

then ψ is radial, i.e. $\psi(r, z) = \psi(r)$.

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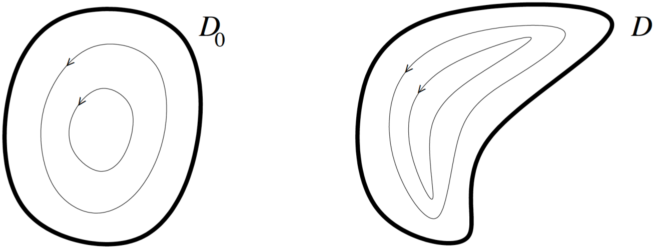
Problem: Given a solution $\boldsymbol{u}_0 = \nabla^\perp \psi_0$ of the steady 2D Euler equations

$$\Delta \psi_0 = \omega_0(\psi_0)$$

for some vorticity $\omega_0 := \omega_0(\psi_0)$ on a domain D_0 and a "nearby" domain D, find a solution $\boldsymbol{u} = \nabla^\perp \psi$ with possibly different vorticity $\omega(\psi)$.



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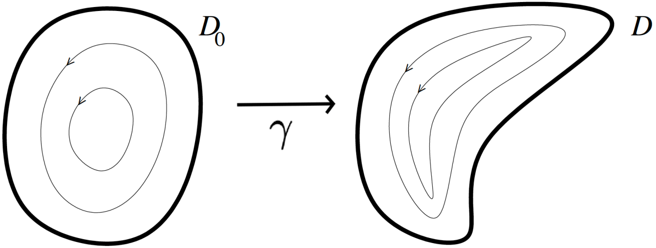
Idea: Seek solution of the form $\psi = \psi_0 \circ \gamma^{-1}$ for a diffeomorphism $\gamma : D_0 \to D$.



Following Vanneste-Wirosoetisno (2005), write $\gamma = \mathit{Id} + \nabla\eta + \nabla^{\perp}\phi$. The η is determined from fixing $\rho = \det \nabla\gamma$ constrained to satisfy $\int_{D_0} \rho = \mathit{Vol}(D)$:

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The other component of the diffeomorphism is fixed by demanding

$$\Delta\psi = \omega(\psi),$$

which, upon substituting $\psi = \psi_0 \circ \gamma^{-1}$, becomes an equation of the form

$$\Big(\Delta - \omega_0'(\psi_0)\Big)\partial_s\phi = \rho^2\omega(\psi_0) - \omega_0(\psi_0) + \mathcal{N}_2(\partial^2\phi, \partial^2\eta),$$

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Hypothesis 1 (H1): The following problem admits only the trivial solution.

$$\Big(\Delta - \omega_0'(\psi_0)\Big)u = 0 \qquad \text{in } D_0,$$
$$u = 0 \qquad \text{on } \partial D_0.$$

Sufficient condition: *Arnol'd stability!* i.e. $\omega_0' > -\lambda_1$ where $\lambda_1 > 0$ is the smallest eigenvalue of $-\Delta$ in D_0 with homogeneous boundary conditions.

Then γ is found by solving a nonlinear elliptic system for $v := \partial_s \phi$ and η

$$\Delta \eta = \rho - 1 + \mathcal{N}_1(\partial^2 \phi, \partial^2 \eta),$$
$$(\Delta - \omega_0'(\psi_0)) v = \omega(\psi_0) - \omega_0(\psi_0) + \mathcal{N}_2(\partial^2 \phi, \partial^2 \eta, 1 - \rho),$$

Boundary conditions that $\gamma : \partial D_0 \to \partial D$ that translate to Dirichlet condition for v and a Neumann condition for η.

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In order to recover ϕ from v, one uses ω. Specifically, inverting $\Delta - \omega_0'(\psi_0)$,

$$v = \Big(\Delta - \omega_0'(\psi_0)\Big)^{-1}_{\mathrm{hbc}}\Big(\omega(\psi_0) - \omega_0(\psi_0) + \mathcal{N}_2\Big).$$

Note that if $v = \partial_s\phi$ for some periodic function ϕ on streamline (dividing by $|\nabla\psi_0|$ and integrating in arc-length), then its integral must vanish. We require

Hypothesis 2 (H2): There exists a constant $C > 0$ such that for all c in the range of ψ_0 the particle travel time on streamlines is bounded

$$\mu(c) = \oint_{\{\psi_0=c\}} \frac{\mathrm{d}\ell}{|\nabla\psi_0|} \leq C, \qquad c \in \mathrm{rang}(\psi_0).$$

Integrating over streamlines, to have $v = \partial_s \phi$ we must have

$$0 = \oint_{\psi_0} v \mathrm{d}s = (K_{\psi_0}\omega)(\psi_0) - (K_{\psi_0}(\omega_0 - \mathcal{N}_2))(\psi_0), \tag{2}$$

where we have introduced $K_{\psi_0} : C^{k-2,\alpha}(I) \to C^{k,\alpha}(I)$ where $I = \mathrm{im}(\psi_0)$

$$(K_{\psi_0} u)(c) := \frac{1}{\mu(c)} \oint_{\{\psi_0 = c\}} \Big(\Delta - \omega_0'(\psi_0)\Big)^{-1}_{\mathrm{hbc}} [u \circ \psi_0] \, \frac{\mathrm{d}\ell}{|\nabla \psi_0|}.$$

We need a hypothesis to choose $\omega := \omega(\psi_0)$ to make (2) hold true, i.e.

$$(K_{\psi_0}\omega)(\psi_0) = (K_{\psi_0}(\omega_0 - \mathcal{N}_2))(\psi_0).$$

Integrating over streamlines, to have $v = \partial_s \phi$ we must have

$$0 = \oint_{\psi_0} v \mathrm{d}s = (K_{\psi_0}\omega)(\psi_0) - (K_{\psi_0}(\omega_0 - \mathcal{N}_2))(\psi_0), \tag{2}$$

where we have introduced $K_{\psi_0} : C^{k-2,\alpha}(I) \to C^{k,\alpha}(I)$ where $I = \mathrm{im}(\psi_0)$

$$(K_{\psi_0} u)(c) := \frac{1}{\mu(c)} \oint_{\{\psi_0 = c\}} \Big(\Delta - \omega_0'(\psi_0)\Big)^{-1}_{\mathrm{hbc}} [u \circ \psi_0] \frac{\mathrm{d}\ell}{|\nabla \psi_0|}.$$

We need a hypothesis to choose $\omega := \omega(\psi_0)$ to make (2) hold true, i.e.

$$(K_{\psi_0}\omega)(\psi_0) = (K_{\psi_0}(\omega_0 - \mathcal{N}_2))(\psi_0).$$

Hypothesis 3 (H3): Fix $k \geq 2$, and let $I = \mathrm{im}(\psi_0)$. For any $g \in C^{k,\alpha}(I)$ such that $g(\psi_0(\partial D_0)) = 0$, there exists a $u \in C^{k-2,\alpha}(I)$ such that $K_{\psi_0} u = g$.

Hypothesis 3 (**H3**) follows if we adopt the slightly stronger hypothesis (**H1**):

Hypothesis 1′ (H1′): The operator $(\Delta - \omega_0'(\psi_0))$ is positive definite, i.e. $\forall f \in H_0^1(D_0)$ there exists $C > 0$ such that $\langle (\Delta - \omega_0'(\psi_0))\, f, f \rangle_{L^2} \geq C \| f \|_{H^1}^2$.

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The idea behind (**H1′**) $\implies$ (**H3**) is that $K_{\psi_0} u := \mathbb{P}_{\psi_0}(\Delta - \omega_0'(\psi_0))^{-1}_{\mathrm{hbc}}[u]$ where

$$(\mathbb{P}_{\psi_0} f)(c) := \frac{1}{\mu(c)} \oint_{\{\psi_0 = c\}} f \frac{\mathrm{d}\ell}{|\nabla\psi_0|}, \qquad \text{for all } c \in \mathrm{im}(\psi_0)$$

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is a projection on L^2. Checked by calculation in action-angle coordinates. Then, in a Hilbert space H, if P is a projection and A is bounded positive operator then the compression PAP is positive in PH since

$$\langle PAPx, x \rangle_H = \langle APx, Px \rangle_H \geq C \langle Px, Px \rangle_H.$$

A strictly positive bounded operator in L^2 like $(\Delta - \omega_0'(\psi_0))^{-1}$ remains positive after compression. Thus the operator PA is invertible from $PH \to PH$.

Theorem (Constantin-Drivas-G.): Let $D_0 \subset \mathbb{R}^2$ with smooth boundary ∂D_0. Suppose $\psi_0 \in C^{k,\alpha}(D_0)$ for some $\alpha > 0$, $k \geq 2$ satisfies $\Delta\psi_0 = \omega_0(\psi_0)$ for some $\omega_0 \in C^{k-2,\alpha}(\mathbb{R})$. Suppose (**H1**), (**H2**) and (**H3**) and that $\int_{D_0} \rho = VolD$. Then there are $\varepsilon_1, \varepsilon_2$ depending only on D_0, ω_0 and $\|\psi_0\|_{C^{k,\alpha}}$ such that if

$$\|\partial D - \partial D_0\|_{C^{k,\alpha}(\mathbb{R})} \leq \varepsilon_1,$$
$$\|1 - \rho\|_{C^{k,\alpha}(D_0)} \leq \varepsilon_2,$$

there is a diffeomorphism $\gamma : D_0 \to D$ with Jacobian $\det(\nabla\gamma) = \rho$, and a function $\omega : \mathbb{R} \to \mathbb{R}$ so that $\psi = \psi_0 \circ \gamma^{-1} \in C^{k,\alpha}(D)$ and ψ satisfies $\Delta\psi = \omega(\psi)$. Thus, $u = \nabla^{\perp}\psi$ is an Euler solution in D nearby u_0.

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REMARK: (**H1′**) is satisfied and thus so is (**H3**) for Arnol'd stable solution:

$$-\lambda_1 < \omega_0' < 0, \qquad \text{or} \qquad 0 < \omega_0' < \infty.$$

The condition is open, so are nearby deformations are Arnol'd stable,

Arnol'd stable solutions are non-isolated and structurally stable.

Return to MHS

Consider again

$$\begin{aligned} \operatorname{curl} B \times B = \nabla P, &\quad \text{in } T, \\ \nabla \cdot B = 0, &\quad \text{in } T, \\ B \cdot \hat{n} = 0, &\quad \text{on } \partial T. \end{aligned}$$

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$$\begin{aligned} \partial_R^2 \psi + \partial_Z^2 \psi - \frac{1}{R} \partial_R \psi + R^2 P'(\psi) + CC'(\psi_0) = 0, &\qquad \text{in } D, \\ \psi = \text{const.} &\qquad \text{on } \partial D \end{aligned}$$

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Are there any other "symmetric" solutions with flux functions?

Quasisymmetry

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Let ξ be a non-vanishing vector field tangent to ∂T. We say B is *quasisymmetric* with respect to ξ if there is a function ψ with $|\nabla\psi| > 0$ satisfying

$$\operatorname{div}\xi = 0$$

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The second point implies

$$B \cdot \nabla\psi = \xi \cdot \nabla\psi = 0$$

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The magnetic differential equation is

$$\begin{aligned} B \cdot \nabla u &= -p'(\psi)(B \times \nabla\psi) \cdot \nabla |B|^{-2} \\ &= -p'(\psi) \left(C(\psi)\xi \times \nabla\psi \cdot \nabla |B|^{-2} + |\nabla\psi|^2 \xi \cdot \nabla |B|^{-2} \right). \end{aligned} \tag{3}$$

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$$\partial_\theta u + \iota(\psi) \partial_\phi u = c(\psi) \partial_\theta f,$$

so Grad's argument does *not* rule out these solutions.

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Even so, there are no known examples of this type!

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Let $\pi(X, Y) = \nabla_X \xi \cdot Y + \nabla_Y \xi \cdot X$ denote the deformation tensor of ξ.

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$$\pi(\xi, \xi) + \frac{2}{C(\psi)}\pi(\xi, \xi \times \nabla\psi) + \frac{1}{C(\psi)^2}\pi(\xi \times \nabla\psi, \xi \times \nabla\psi) = 0.$$

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When ξ is a Killing field, these are trivial!

To satisfy MHS ψ needs to satisfy the *quasisymmetric Grad-Shafranov equation*

$$\Delta\psi - \frac{\xi \times \operatorname{curl}\xi}{|\xi|^2} \cdot \nabla\psi + \frac{\xi \cdot \operatorname{curl}\xi}{|\xi|^2} C(\psi) + CC'(\psi) + |\xi|^2 P'(\psi) = 0.$$

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It is not clear if this is even consistent with solutions satisfying $\xi \cdot \nabla\psi = 0$!

Application of the deformation theorem.

Return to magnetohydrostatics & stellarator confinement fusion,

Theorem (Constantin-D.rivas-G) There exist approximate quasisymmetric MHS solutions with flux functions provided they are sustained by forcing f with $|f| \lesssim |\xi - \xi_0|$ where ξ_0 is the nearest Euclidean Killing field to ξ and ξ is the symmetry direction.

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Idea of proof:
Choose a metric g for which a given ξ *does* generate an isometry.
Look for a solution of the form

$$B_g = \frac{1}{|\xi|_g^2}\left(C(\psi)\xi + \sqrt{|g|}\xi \times_g \nabla_g \psi\right)$$

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Surprisingly

$$\operatorname{div} B_g = 0, \qquad B \times \xi = \nabla \psi.$$

Require that B_g satisfies the MHS equations with respect to the metric g,

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This gives a two-dimensional Grad-Shafranov equation

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which is consistent with $\mathcal{L}_\xi \psi = 0$!

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Solve this by deforming a solution to the axisymmetric Grad-Shafranov equations. The resulting field satisfies the usual MHS equations up an error controlled by $|\xi - \xi_0|$. Also satisfies the third constraint of quasisymmetry to the same order.

Thanks for your attention!