Construction of approximate quasisymmetric equilibria sustained by a small force

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Magnetohydrostatic Equilbria

Let $T \subset \mathbb{R}^3$ be a domain with smooth boundary (e.g. the infinite cylinder or the axisymmetric torus). The Magnetohydrostatic (MHS) equations in T read

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\begin{aligned}
J \times B &= \nabla P + f, && \text{in } T, \\
\nabla \cdot B &= 0, && \text{in } T, \\
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where $J = \nabla \times B$ is the current, f is an external force and P is the 'plasma pressure'.

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PROGRAM: Identify and construct (smooth) magnetohydrostatic equilibria which are effective at confining ions during a nuclear fusion reaction.

Quasisymmetric Equilibrium in Stellarator Geometry

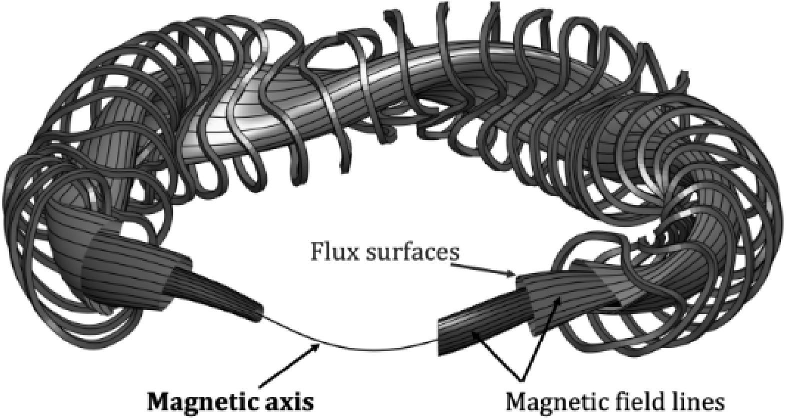

Figure taken from Landreman (2019).

Definitions of Quasisymmetric Equilibria

Definition[Rodríguez, Helander, Bhattacharjee 2020 (preprint)]:
Let ξ be a non-vanishing vector field tangent to ∂T. We say that ξ is a *weak quasisymmetry* and the field B is *weakly quasisymmetric* if

$$\operatorname{div} \xi = 0, \tag{1}$$

$$\xi \times B = -\nabla \psi, \tag{2}$$

$$\xi \cdot \nabla |B| = 0, \tag{3}$$

for some flux function $\psi : T \to \mathbb{R}$.

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Definition[Burby-Kallinikos-MacKay 2019, Landreman 2019]: Let ξ be a non-vanishing vector field tangent to ∂T. We say that ξ is a *strong quasisymmetry* and B is *strongly quasisymmetric* if condition (3) is replaced by

$$\xi \times J = \nabla (B \cdot \xi), \tag{4}$$

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By a result of Burby-Kallinikos-MacKay (2019), in strong quasisymmetry B must be of the form

$$B = \frac{1}{|\xi|^2}\big(C(\psi)\xi + \xi \times \nabla\psi\big) \tag{5}$$

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QUESTION: When does the ansatz (5) satisfy (1)–(3) and MHS?

Quasisymmetric Equilibria

The conditions for quasisymmetry are closely related to deformation tensor $\mathcal{L}_\xi \delta$

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Proposition: Let ξ be a non-vanishing and divergence-free, ψ be such that $\xi \cdot \nabla\psi = 0$ and $|\nabla\psi| > 0$, and B be as in (5). Then:
The field B is divergence-free if and only if

$$(\mathcal{L}_\xi \delta)(\xi, \nabla^\perp\psi) = -C(\psi)(\mathcal{L}_\xi \delta)(\xi, \xi), \qquad \nabla^\perp = \xi \times \nabla. \tag{7}$$

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Condition (4) required for strong quasisymmetry is satisfied if and only if

$$(\mathcal{L}_\xi \delta)(\nabla^\perp \psi, \nabla^\perp \psi) = C^2(\psi)(\mathcal{L}_\xi \delta)(\xi, \xi), \tag{9}$$

$$(\mathcal{L}_\xi \delta)(\nabla \psi, \nabla \psi) = -|B|^2(\mathcal{L}_\xi \delta)(\xi, \xi), \tag{10}$$

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If ξ is a Killing field for the Euclidean metric, then $\mathcal{L}_\xi \delta \equiv 0$ and all the conditions (7)–(11) are satisfied independent of the nature of ψ and $C(\psi)$.

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Proposition: Let ξ be a non-vanishing and divergence-free vector field tangent to ∂T, ψ be such that $\xi \cdot \nabla\psi = 0$ and $|\nabla\psi| > 0$, and B be given by (5). Then B is a **weakly** quasisymmetric solution of MHS with $C = B \cdot \xi$ constant on flux surfaces if and only if (7)–(8) hold, and

$$\Delta\psi + CC'(\psi) - \frac{1}{|\xi|^2}\left[\xi \times \operatorname{curl}\xi \cdot \nabla\psi - C(\psi)\xi \cdot \operatorname{curl}\xi\right] + |\xi|^2 P'(\psi) = C(\psi)\frac{(\mathcal{L}_\xi\delta)(\nabla\psi, B)}{|\nabla\psi|^2} - |\xi|^2 \frac{f \cdot \nabla\psi}{|\nabla\psi|^2},$$

$$-\frac{|B|^2}{|\xi|^2} C(\psi)(\mathcal{L}_\xi\delta)(\xi, \xi) = f \cdot \nabla^\perp\psi,$$

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Proposition:Let ξ be a non-vanishing and divergence-free vector field tangent to ∂T, ψ be such that $\xi \cdot \nabla\psi = 0$ and $|\nabla\psi| > 0$, and B be given by (5). Then B is a **strongly** quasisymmetric solution of MHS with $C = B \cdot \xi$ constant on flux surfaces if and only if (7)–(8) hold and

$$\Delta\psi + CC'(\psi) - \frac{1}{|\xi|^2}\left[\xi \times \operatorname{curl}\xi \cdot \nabla\psi - C(\psi)\xi \cdot \operatorname{curl}\xi\right] + |\xi|^2 P'(\psi) = |\xi|^2 \frac{f \cdot \nabla\psi}{|\nabla\psi|^2},$$

$$f \cdot \nabla^{\perp}\psi = 0,$$

$$f \cdot \xi = 0.$$

This generalized Grad-Shafranov (gGS) equation for ψ was derived by Burby-Kallinikos-MacKay (2019). The condition $\xi \cdot \nabla\psi = 0$ is non-trivial!

Constraints on the deformation tensor with no forcing

Proposition: If ξ is a **weak** quasisymmetry for B then the deformation tensor takes the form

$$(\mathcal{L}_\xi \delta)_{\mathcal{B}} = \begin{pmatrix} 0 & \mathcal{L}_\xi \delta(\nabla\psi, \nabla^\perp\psi) & (\mathcal{L}_\xi \delta)(\widehat{\nabla\psi}, \widehat{\xi}) \\ \mathcal{L}_\xi \delta(\nabla\psi, \nabla^\perp\psi) & 0 & 0 \\ (\mathcal{L}_\xi \delta)(\widehat{\nabla\psi}, \widehat{\xi}) & 0 & 0 \end{pmatrix}_{\mathcal{B}} \tag{12}$$

where the matrix $(\mathcal{L}_\xi \delta)$ is represented in the orthonormal basis $\mathcal{B} := \{\widehat{\nabla\psi}, \widehat{\nabla^\perp\psi}, \widehat{\xi}\}$.

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If ξ is a **strong** quasisymmetry for B then the deformation tensor takes the form

$$(\mathcal{L}_\xi\delta)_{\mathcal{B}} = (\mathcal{L}_\xi\delta)(\widehat{\nabla\psi}, \widehat{\xi}) \begin{pmatrix} 0 & -\frac{C(\psi)}{|\nabla\psi|} & 1 \\ -\frac{C(\psi)}{|\nabla\psi|} & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}_{\mathcal{B}} \tag{13}$$

QUESTION: Given the many constraints on (ξ, B) (conditions (7)–(11) and a "ξ-independent" solution of gGS), are there any examples?

Examples of Quasisymmetry: helical symmetry in an infinite cylinder

Consider the helical vector field defined by

$$\xi_0 = \ell e_z - mre_\theta,$$

whose integral curves generate the infinite cylinder

$$T_0 = \{(r, \theta, z) \in (0, 1] \times \mathbb{T} \times \mathbb{R}\}.$$

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Then ξ_0 is a Killing field so all the conditions for quasisymmetry are satisfied. The flux function is determined by the helical Grad-Shafranov

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\ell^2 + m^2 r^2}\frac{\partial}{\partial r}\psi\right) + \frac{1}{r^2}\frac{\partial^2}{\partial u^2}\psi + P'(\psi) + \frac{CC'(\psi)}{\ell^2 + m^2 r^2} - \frac{2m\ell C(\psi)}{(\ell^2 + m^2 r^2)^2} = 0,$$

with helical coordinate $u = \ell\theta + mz$. Since the coefficients of this equation are independent of $v = \ell z - m\theta$, it admits solutions with $\xi_0 \cdot \nabla\psi = 0$.

For any solution of the helical Grad-Shafranov equation, B_0 defined by

$$B_0 = \frac{1}{|\xi_0|^2}\big(C(\psi)\xi_0 + \xi_0 \times \nabla\psi\big),$$

is automatically a quasisymmetric MHS equilibrium on the straight cylinder T_0.

Examples of Quasisymmetry: axisymmetry in solid torus

Consider the Killing vector field defined by

$$\xi_0 = Re_\phi$$

whose integral curves are periodic and generate the axisymmetric torus with axis $R = R_0$,

$$T_0 = \{(R, Z, \phi) | R = R_0 + r\cos\theta, Z = r\sin\theta, r \in [0, 1], \theta \in [0, 2\pi], \phi \in [0, 2\pi]\}.$$

The flux function is determined by the toroidal Grad-Shafranov equation

$$\partial_r^2\psi + \frac{1}{r^2}\partial_\theta^2\psi + \frac{1}{r}\partial_r\psi - \frac{1}{R}\left(\cos\theta\partial_r\psi - \frac{\sin\theta}{r}\partial_\theta\psi\right) + R^2P'(\psi) + CC'(\psi) = 0,$$

with $R = R_0 + r\cos\theta$. Since the coefficients of this equation are independent of ϕ, it admits solutions with $\xi_0 \cdot \nabla\psi = 0$.

Nonexistence outside of symmetry? Grad's conjecture

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We formalize a version of this statement as a rigidity property of equilibria

> **Conjecture** (Grad, 1967): Any non-isolated and non-vanishing smooth MHS equilibrium on a domain $T \subset \mathbb{R}^3$ (diffeomorphic to the solid cylinder or torus) which has a pressure p possessing nested level sets which foliate T is either axially or helically symmetric.

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This conjecture remains open. However a natural question is

QUESTION: If one relaxes some of the requirements of quasisymmetry, is it possible to construct non-symmetric equilibrium states of plasma?

Main Results: Approximate quasisymmetry on cylindrical domain

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Theorem (C-D-G, in prep): There exists ξ on $\mathbb{R}^3$ which is close to axisymmetric (i.e. with $|\xi - e_z| = O(\varepsilon)$ for ε small) whose integral curves are periodic and generate a domain T close to the **straight cylinder**, with the property that there is a vector field $B : T \to \mathbb{R}^3$ solving

$$\begin{aligned} J \times B &= \nabla P + f, && \text{in } T, \\ \nabla \cdot B &= 0, && \text{in } T, \\ B \cdot \hat{n} &= 0, && \text{on } \partial T, \end{aligned}$$

with an explicit force f which is $O(\varepsilon)$. Moreover, ξ is an approximate quasisymmetry in the sense that

$$\begin{aligned} \operatorname{div} \xi &= 0, && \text{in } T, \\ \operatorname{curl}(\xi \times B) &= 0, && \text{in } T, \\ \xi \times J &= \nabla(B \cdot \xi) + O(\varepsilon), && \text{in } T, \end{aligned}$$

and there exists a flux function ψ with nested level surfaces such that

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Main Results: Approximate quasisymmetry on a toroidal domain

Theorem (C-D-G, in prep): There exists ξ on $\mathbb{R}^3$ which is close to axisymmetric (i.e. with $\mathcal{L}_\xi \delta = O(\varepsilon)$ for ε small) whose flow generates a domain T close to the **axisymmetric torus** with large aspect ratio (i.e. with $(\min \text{radius})/(\max \text{radius}) = O(1/R)$ for R large) such that there is a vector field $B : T \to \mathbb{R}^3$ solving

$$\begin{aligned} J \times B &= \nabla P + f, && \text{in } T, \\ \nabla \cdot B &= 0, && \text{in } T, \\ B \cdot \hat{n} &= 0, && \text{on } \partial T, \end{aligned}$$

with an explicit force f which is $O(\max\{\varepsilon, 1/R\})$. Moreover, ξ is an approximate quasisymmetry in the sense that

$$\begin{aligned} \operatorname{div} \xi &= 0, && \text{in } T, \\ \operatorname{curl}(\xi \times B) &= 0, && \text{in } T, \\ \xi \times J &= \nabla(B \cdot \xi) + O(\max\{\varepsilon, 1/R\}), && \text{in } T, \end{aligned}$$

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Ideas of the proof:

Basic technique due to Vanneste-Wirosoetisno (2005).

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Problem: Given a solution $\boldsymbol{u}_0 = \nabla^\perp \psi_0$ of the steady 2D Euler equations with vorticity ω_0,

$$\Delta \psi_0 = \omega_0(\psi_0) \tag{14}$$

on a domain D_0 and a "nearby" domain D, find a solution $\boldsymbol{u} = \nabla^\perp \psi$ with possibly different vorticity $\omega(\psi)$.

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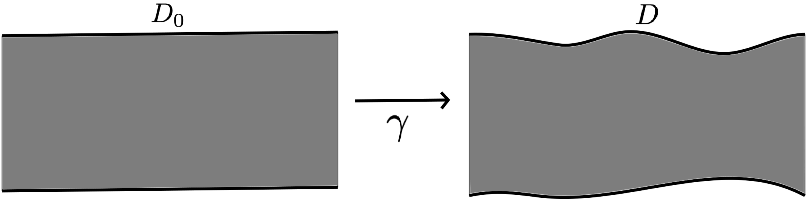
Idea: look for a solution of the form $\psi = \psi_0 \circ \gamma^{-1}$ for a diffeomorphism $\gamma : D_0 \to D$.



Ideas of the proof:

Writing $\gamma = \mathit{Id} + \nabla\eta + \nabla^{\perp}\phi$, the requirement that

$$\Delta\psi = \omega(\psi_0), \tag{15}$$

becomes an equation of the form

$$\Delta\phi = F(\partial^2\phi, \partial^2\eta), \tag{16}$$

The function η is determined from the requirement that $\mathit{Vol}(D) = \mathit{Vol}(D_0)$, since

$$1 = \det \nabla\gamma = 1 + \Delta\eta + G(\partial^2\phi, \partial^2\eta). \tag{17}$$

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Then γ can be found by solving a nonlinear system of elliptic equations

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Observation: Trivial modification to allow for $Vol(D) \sim Vol(D_0)$ by picking a function ρ with $\int_{D_0} \rho = Vol(D)$ and solving

$$\Delta\eta = 1 - \rho + G(\partial^2\phi, \partial^2\eta). \tag{20}$$

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Step 1: Take an solution defined on the cylinder T_0 which is a function only of r, i.e. $\psi_0 := \psi_0(r)$, having the property

$$\left| \frac{C(\psi_0)}{\psi_0'} \right| \approx \frac{1}{\epsilon_0}. \tag{21}$$

Step 2: Let ξ be a vector field with a periodic flow which is nearly axisymmetric and generates a 'nearby' non-axisymmetric cylinder T. We require a special relationship between components of its deformation tensor

$$\frac{|(\mathcal{L}_\xi \delta)(\xi, \xi)|}{|(\mathcal{L}_\xi \delta)(\xi, \xi \times e_r)|} \approx \epsilon_0 > 0. \tag{22}$$

Ideas of the proof:

Step 3: Fix coordinates (x_1, x_2, x_3) on $\mathbb{R}^3$ so that $\xi \cdot \nabla = \frac{\partial}{\partial x_3}$ and a disk D in the $x_3 = 0$ plane. We will construct a solution in the cylinder T generated from the integral curves of ξ starting from D.

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Step 4: In these coordinates (gGS) takes the form

$$\Delta\psi + G(x_1, x_2, x_3, \psi) = 0$$

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for an explicit function G. Freeze coefficients and write as

$$\Delta_{2d}\psi + G_1(x_1, x_2, \psi) + L\psi + R(x_1, x_2, x_3, \psi) = 0 \tag{23}$$

where Δ_{2d} is the part of the Laplacian only involving derivatives in the x_1, x_2 direction, where $L = L(\partial_{x^3})$ and the remainder R satisfies

$$|R| \lesssim |\mathcal{L}_\xi \delta| = O(\varepsilon). \tag{24}$$

Ideas of the proof:

Step 5: Construct a streamfunction ψ with nested level sets foliating the domain and enjoying the property that $\xi \cdot \nabla\psi = 0$ (i.e. $\psi = \psi(x_1, x_2)$ in this coordinate system), satisfying

$$\Delta_{2d}\psi + G_1(x_1, x_2, \psi) = 0. \tag{25}$$

This ψ nearly satisfies (gGS) in the sense that

$$|\Delta\psi + G| = O(\varepsilon). \tag{26}$$

This, and the assumption $|\xi - e_z| = O(\varepsilon)$, guarantees that B satisfies MHS with small force $f = O(\varepsilon)$.

Ideas of the proof:

Step 5 (cont.): We do this by looking for ψ of the form $\psi = \psi_0 \circ \gamma^{-1}$ for a diffeomorphism $\gamma = \mathit{Id} + \nabla\eta + \nabla^{\perp}\phi$ for small η, ϕ to be determined. Such a solution automatically has nested level sets foliating the cylinder. The requirement that ψ satisfy (25) becomes a nonlinear elliptic equation for ϕ of the form

$$\Delta\phi = N(\partial^2\phi, \partial^2\eta), \tag{27}$$

for a given nonlinearity N.

Ideas of the proof:

Step 5 (cont.): We do this by looking for ψ of the form $\psi = \psi_0 \circ \gamma^{-1}$ for a diffeomorphism $\gamma = \mathit{Id} + \nabla\eta + \nabla^{\perp}\phi$ for small η, ϕ to be determined. Such a solution automatically has nested level sets foliating the cylinder. The requirement that ψ satisfy (25) becomes a nonlinear elliptic equation for ϕ of the form

$$\Delta\phi = N(\partial^2\phi, \partial^2\eta), \tag{27}$$

for a given nonlinearity N.

Step 6: We must find a solution ψ consistent with $\mathrm{div}\, B = 0$, i.e. so that

$$C(\psi)(\mathcal{L}_\xi\delta)(\xi,\xi) + (\mathcal{L}_\xi\delta)(\xi, \xi \times \nabla\psi) = 0. \tag{28}$$

We emphasize that any deviation from (28) holding exactly cannot be compensated directly by a force as the condition $\mathrm{div}\, B = 0$ sees the form of B alone and can be altered only through changing ψ.

Ideas of the proof:

Step 5 (cont.): We do this by looking for ψ of the form $\psi = \psi_0 \circ \gamma^{-1}$ for a diffeomorphism $\gamma = \mathit{Id} + \nabla\eta + \nabla^{\perp}\phi$ for small η, ϕ to be determined. Such a solution automatically has nested level sets foliating the cylinder. The requirement that ψ satisfy (25) becomes a nonlinear elliptic equation for ϕ of the form

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A calculation shows that (28) is

$$C(\psi)(\mathcal{L}_\xi\delta)(\xi, \xi) + \psi_0'(\det\nabla\gamma)^{-1}\big((\mathcal{L}_\xi\delta)(\xi, \xi \times e_r) + (\mathcal{L}_\xi\delta)(\xi, \xi_{\phi\eta})\big) \tag{29}$$

with $\xi_{\phi\eta} \sim (\partial^2\phi, \partial^2\eta, 0)$.

Ideas of the proof:

Step 6 (cont.): Therefore $\mathrm{div}\, B = 0$ provided we can choose γ satisfying

$$(\det \nabla\gamma)^{-1} = \frac{C(\psi_0 \circ \gamma^{-1})}{\psi_0' \circ \gamma^{-1}} \frac{(\mathcal{L}_\xi \delta)(\xi, \xi)}{(\mathcal{L}_\xi \delta)(\xi, \xi \times e_r) + (\mathcal{L}_\xi \delta)(\xi, \xi_{\phi\eta})}, \tag{30}$$

Ideas of the proof:

Step 6 (cont.): Therefore $\operatorname{div} B = 0$ provided we can choose γ satisfying

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In general we have

$$\Delta\eta = (\det \nabla\gamma)^{-1} - 1 + N(\partial^2\eta, \partial^2\phi), \tag{31}$$

and by our assumptions

$$(\det \nabla\gamma)^{-1} - 1 \sim \epsilon_0(\partial^2\phi + \partial^2\eta). \tag{32}$$

Ideas of the proof:

Step 6 (cont.): Therefore $\operatorname{div} B = 0$ provided we can choose γ satisfying

$$(\det \nabla\gamma)^{-1} = \frac{C(\psi_0 \circ \gamma^{-1})}{\psi_0' \circ \gamma^{-1}} \frac{(\mathcal{L}_\xi \delta)(\xi, \xi)}{(\mathcal{L}_\xi \delta)(\xi, \xi \times e_r) + (\mathcal{L}_\xi \delta)(\xi, \xi_{\phi\eta})}, \tag{30}$$

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and by our assumptions

$$(\det \nabla\gamma)^{-1} - 1 \sim \epsilon_0(\partial^2\phi + \partial^2\eta). \tag{32}$$

Step 7: We then need to solve a system of the form

$$\Delta\eta = \epsilon_0(\partial^2\phi + \partial^2\eta) + F(\partial^2\eta, \partial^2\phi), \tag{33}$$

$$\Delta\phi = F(\partial^2\phi, \partial^2\eta). \tag{34}$$

This can be solved by iteration:

$$\Delta\eta^{N+1} = \epsilon_0(\partial^2\phi^N + \partial^2\eta^N) + F(\partial^2\eta^N, \partial^2\phi^N), \tag{35}$$

$$\Delta\phi^{N+1} = F(\partial^2\phi^N, \partial^2\eta^N). \tag{36}$$

Some final remarks

• The proof is constructive and provides an algorithm which, in principle, can be used to generate these equilibria on the computer.

• The technique is robust to small perturbations, allowing steady states occupying a given domain to be deformed to fit nearby ones for a variety of model equations including 2d Euler, Boussinesq, as well as MHS.

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